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Nonlinear free vibration of conservative oscillators with inertia and static type cubic nonlinearities using homotopy analysis method

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Abstract

In this study, an accurate analytical solution for the nonlinear free vibration of a conservative oscillator with inertia and static type cubic nonlinearities is derived. This solution has been obtained using homotopy analysis method (HAM). Then, homotopy Pade technique is applied to accelerate the convergence rate of the series solution. This study shows that the HAM leads to an accurate analytical solution, which is valid for a wide range of considered system parameters. Unlike the other analytical methods, HAM can control and adjust the convergence region and rate of the approximation series solution. The excellent accuracy of the current results is demonstrated by comparing with the available analytical and numerical results.

1. Introduction

Many physical phenomena are modeled by nonlinear differential equations. As an example, vibration of mechanical systems associated with nonlinear properties can be mentioned. Therefore, the study on the various methods used for solving the nonlinear differential equations is a very important topic for the analysis of engineering practical problems. There are a number of approaches for solving nonlinear equations, which range from completely analytical to completely numerical ones. Besides all advantages of using numerical methods, closed form solutions appear more appealing because they reveal physical insights through the physics of the problem. Also, parametric studies become more convenient with applying analytical methods. Moreover, analytical solutions are generally required for the validation of numerical methods and computer softwares.

Traditional analytical methods, which have been widely used for nonlinear equations include perturbation methods such as the Lindstedt–Poincare, multiple time scales methods and the generalized averaging method

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of Krylov–Bogoliubov–Mitropolski [1–3]. The effectiveness of perturbation methods is limited, because these methods are applicable primarily only when the nonlinear terms in the equation are small relative to the linear terms. The other available analytical methods, which have been used for solving both weakly and strongly nonlinear equations, are the harmonic balance [4,5], equivalent linearization [6], describing function [7] and power series method. These methods have their own limitations and any variation in system parameters can lead to not only quantitative but also qualitative errors in the predicted response. For example, although the power series is a powerful method and has been employed with some successes, but since the method requires the generation of a coefficient for each term in the series, it is relatively tedious and difficult to demonstrate that the series converges [8,9].

The harmonic balance (HB) [1,10,11] is another method for determining analytical approximations to the periodic solutions of differential equations. Although this method is valuable and can solve strongly nonlinear vibration problems, it is usually very difficult to construct higher order of approximations to the solution. This happens because for higher-order approximations, sets of complicated nonlinear algebraic equations must be solved.

In general, both of the perturbation and non-perturbation methods cannot adjust the convergence region and rate of the given approximate series [12,13]. In order to overcome the limitations of traditional analytical methods, homotopy analysis method (HAM) was developed by Liao [12], which has the following advantages over above-mentioned methods:

1. HAM can adjust the convergence region and the rate of approximation series.
2. HAM is easy-to-use analytic tool for solving strongly nonlinear differential equations.

The effectiveness and accuracy of the HAM have been demonstrated in the analysis of some nonlinear problems [14–21]. It is noted that the other powerful analytic techniques for strongly nonlinear problems such as Adomian's decomposition method, Lyapunov's artificial small parameter method, and the δ-expansion method are special cases of the HAM, so that they can be unified in the frame of the HAM [13].

The main objective of present study is to obtain highly accurate analytical solutions for free vibrations of a conservative oscillator with inertia and static type cubic nonlinearities. The HAM and homotopy Pade technique are used to find analytical solutions for this problem with the nonlinear governing differential equation. It is shown that the solution is quickly convergent and its components can be simply calculated. Also, compared to other analytical methods, it can be observed that the results of HAM and homotopy Pade technique are accurate and require smaller computational effort. Combination of the HAM and homotopy Pade technique accelerates the convergence of the results. An excellent accuracy of the HAM results indicates that this method can be used for problems in which the strong nonlinearities are taken into account.

2. Governing equation of the problem

Many engineering structures can be modeled as a slender, elastic cantilever beam carrying a concentrated mass at an intermediate point along its span [22,23]. This system can be simulated by a mass with serial linear and nonlinear stiffnesses on a frictionless contact surface as shown in Fig. 1. For this system, the governing differential equation of the motion is in the following form [24,25]:

$$(1+3\varepsilon z v^2)\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}+6\varepsilon z v\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2+\omega_e^2 v+\varepsilon\omega_e^2 v^3=0, \tag{1}$$

$$v(t)=y_2(t)-y_1(t), \tag{2}$$

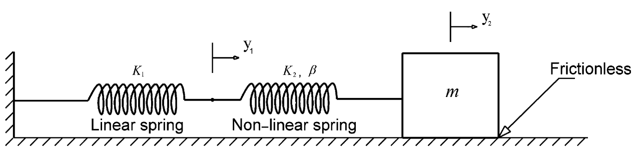

Fig. 1. Schematic of the problem: a mass with serial linear and nonlinear stiffnesses on a friction less contact surface.

where

$$\begin{aligned} \varepsilon &= \frac{\beta}{K_2}, \\ \xi &= \frac{K_2}{K_1}, \\ z &= \frac{\xi}{1+\xi}, \\ \omega_e &= \sqrt{\frac{K_2}{m(1+\xi)}}. \end{aligned} \tag{3}$$

Parameters β, K_2, K_1, v, m and ω_e are nonlinear spring constant, linear portion of the nonlinear spring constant, linear spring constant, deflection of nonlinear spring, mass and the natural frequency, respectively [25]. Subject to the following initial conditions:

$$v(0) = a, \quad \dot{v}(0) = \frac{\mathrm{d}v}{\mathrm{d}t}(0) = 0. \tag{4}$$

Under the transformation $\tau = \omega t$, where ω denotes the frequency of vibration, Eq. (1) takes the following form:

$$\omega^2[(1 + 3\varepsilon z v^2)\ddot{v} + 6\varepsilon z v \dot{v}^2] + \omega_e^2 v + \varepsilon \omega_e^2 v^3 = 0, \tag{5}$$

with the following initial conditions:

$$v(0) = a, \quad \dot{v}(0) = 0. \tag{6}$$

3. Homotopy analysis method

3.1. Basic idea

Homotopy analysis is a general analytic method for solving the nonlinear differential equations [12,13]. The HAM transforms a nonlinear differential equation into an infinite number of linear differential equation with embedding an auxiliary parameter (q) that typically ranges from zero to one [13]. As q increases from zero to one, the solution varies from the initial guess to the exact solution. To illustrate the basic ideas of the HAM, consider a nonlinear differential equation as

$$N[v(t)] = 0, \tag{7}$$

where N is a general nonlinear differential operator, and $v(t)$ is an unknown function of the parameter t. The homotopy is constructed as follows:

$$\overline{H}(\phi, q, \hbar, H(t)) = (1-q)L[\phi(t,q) - v_0(t)] - q\hbar H(t) N[\phi(t,q)], \tag{8}$$

where ϕ, $\hbar$ and $H(t)$ are a function of t and q, the nonzero auxiliary parameter and the nonzero auxiliary function, respectively. The auxiliary parameter and the auxiliary function adjust convergence region of the solution. The parameter L denotes an auxiliary linear operator. As q increases from zero to one, the $\phi(t, q)$ varies from the initial approximation to the exact solution. In other words, $\phi(t, 0) = v_0(t)$ is the solution of the $\overline{H}(\phi, q, \hbar, H(t))|_{q=0} = 0$, and $\phi(t,\ 1) = v_0(t)$ is the solution of the $\overline{H}(\phi, q, \hbar, H(t))|_{q=1} = 0$. Enforcing $\overline{H}(\phi, q, \hbar, H(t)) = 0$, the zero-order deformation is constructed as:

$$(1-q)L[\phi(t,q) - v_0(t)] = q\hbar H(t) N[\phi(t,q)], \tag{9}$$

with the following initial conditions:

$$\phi(0; q) = a, \quad \frac{\mathrm{d}\phi(0,q)}{\mathrm{d}t} = 0. \tag{10}$$

The functions $\phi(t, q)$ and $\omega(q)$ can be expanded as power series of q using Taylor's theorem as

$$\phi(t,q) = \phi(t,0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \phi(t,q)}{\partial q^m}\bigg|_{q=0} q^m = v_0(t) + \sum_{m=1}^{\infty} v_m(t) q^m, \tag{11}$$

$$\omega(q) = \omega_0 + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \omega(q)}{\partial q^m}\bigg|_{q=0} q^m = \omega_0 + \sum_{m=1}^{\infty} \omega_m q^m, \tag{12}$$

where $v_m(t)$ and ω_m are called the m-order deformation derivatives.

Differentiating zero-order deformation equation with respect to q and then setting $q = 0$, yields the first-order deformation equation ($m = 1$) which gives the first-order approximation of the $v(t)$ as follows:

$$L[v_1(t)] = \hbar H(t) N[v_0(t), \omega_0]\big|_{q=0}, \tag{13}$$

with the following initial conditions:

$$v_1(0) = 0, \quad v_1(0) = 0. \tag{14}$$

The higher order approximations of the solution can be obtained by calculating the m-order ($m>1$) deformation equation. The m-order deformation equation can be calculated by differentiating Eqs. (13) and (14) m times with respect to q as follows [14,15]:

$$L[v_m(t) - v_{m-1}(t)] = \hbar H(t) R_m(\vec{v}_{m-1}, \vec{\omega}_{m-1}), \tag{15}$$

where the $\vec{v}_{m-1}$, $\vec{\omega}_{m-1}$ and $R_m(\vec{v}_{m-1}, \vec{\omega}_{m-1})$ are defined as follows:

$$R_m(\vec{v}_{m-1}, \vec{\omega}_{m-1}) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1} N[\phi(t,q), \omega(q)]}{\mathrm{d}q^{m-1}}\bigg|_{q=0}, \tag{16}$$

$$\vec{v}_{m-1} = \{v_0, v_1, v_2, \ldots, v_{m-1}\}, \tag{17}$$

$$\vec{\omega}_{m-1} = \{\omega_0, \omega_1, \omega_2, \ldots, \omega_{m-1}\}, \tag{18}$$

subject to the following initial conditions:

$$v_m(0) = \dot{v}_m(0) = 0. \tag{19}$$

3.2. Homotopy Pade technique

Pade approximant is the best approximation of a function by a rational function of a given order [26,27]. A Pade approximant often gives better approximation of a function than those of Taylor series. Also, Pade approximation may work in cases where the Taylor series does not converge. In order to calculate a Pade approximant of type $[m, n]$, let f be a function expanded in the form of power series as follows:

$$f(z) = \sum_{k=0}^{m+n+1} a_k z^k, \tag{20}$$

where $a_k = f^{(k)}(0)/k!, \quad k = 0, 1, 2, \ldots, m+n+1$. $[m, n]$ Pade approximant of f is represented by a rational function r that can be written in the following form:

$$r(z) = \frac{b_0 + b_1 z + \cdots + b_m z^m}{1 + c_1 z + \cdots + c_n z^n} \equiv \frac{p(z)}{q(z)}. \tag{21}$$

Generally, f is expanded in Taylor (or Laurent) series about the point $x = a$ (if a is not specified then the expansion is about the point $x = 0$), to order $m+n+1$, and then the Pade rational approximation is computed. Pade technique is used to accelerate the convergence of a given series.

The homotopy Pade technique [18] is a combination of the above-mentioned traditional Pade technique with the homotopy analysis method. In order to calculate the $[m, n]$ homotopy Pade approximant of $\phi(t, q)$, first the traditional $[m, n]$ Pade technique is employed about the embedding parameter q:

$$\phi(t;q) = \frac{\sum_{k=0}^{m} A_{m,k}(t)q^k}{\sum_{k=0}^{n} B_{m,k}(t)q^k}, \tag{22}$$

where the coefficients $A_{m,k}(t)$ and $B_{m,k}(t)$ are determined by the first $(m+n)$th-order approximations of $v(t)$. It is noted that using homotopy Pade approximant, the number of order of approximation required to obtain an accurate solution is reduced [27].

3.3. Application of the HAM

Free oscillation of a system without damping is a periodic motion and can be expressed by the following base functions [28]:

$$\{\cos(m\tau), \quad m = 1, 2, 3, \ldots\}. \tag{23}$$

In order to satisfy the initial conditions, the initial guess of $v(\tau)$ for zero-order deformation equation is chosen as follows:

$$v_0(\tau) = a\cos\tau. \tag{24}$$

To construct the homotopy function, the auxiliary linear operator for vibration of a conservative system is selected as [13]

$$L[v(\tau;q)] = \omega_0^2\left(\frac{\partial^2 v(\tau;q)}{\partial\tau^2} + v(\tau;q)\right). \tag{25}$$

The auxiliary linear operator L is chosen in such a way that all solutions of the corresponding high-order deformation equations exist and can be expressed by the general form of the base function. From Eq. (5), the nonlinear operator is written as

$$N[v(\tau;q),\omega] = \omega^2\left[(1+3\varepsilon z(v(\tau;q))^2)\frac{\mathrm{d}^2 v(\tau;q)}{\mathrm{d}\tau^2} + 6\varepsilon z v(\tau;q)\left(\frac{dv(\tau;q)}{d\tau}\right)^2\right] + \omega_e^2 v(\tau;q) + \varepsilon\omega_e^2(v(\tau;q))^3. \tag{26}$$

The solution must comply with the general form of the base functions. Therefore, the auxiliary function $(H(\tau))$ must be chosen as follows:

$$H(\tau) = 1. \tag{27}$$

Due to odd nonlinearity of considered conservative system, it is found that R_m can be expressed by

$$R_m(\tau,\omega_{m-1}) = \sum_{n=0}^{m} \mathrm{d}_n(\omega_{m-1})\cos((2n+1)\tau). \tag{28}$$

According to the property of the linear operator, if the term $\cos(\tau)$ exist in R_m, the secular term $\tau\sin(\tau)$ will appear in the final solution. Therefore, the coefficient of $\cos(\tau)$ in R_m should be equal to zero:

$$d_0(\omega_{m-1}) = 0. \tag{29}$$

Solving Eq. (29), ω_{m-1} is obtained. For the first-order approximation of HAM, R_1 is as follows:

$$R_1 = \left(-\omega_0^2 a - \tfrac{3}{4}\omega_0^2 a^3\varepsilon z + \omega_e^2 a + \tfrac{3}{4}\omega_e^2 a^3\varepsilon\right)\cos(\tau) + \left(\tfrac{1}{4}\omega_e^2 a^3\varepsilon - \tfrac{9}{4}\omega_0^2 a^3\varepsilon z\right)\cos(3\tau). \tag{30}$$

Thus, ω_0 can be written as

$$\omega_0 = \omega_e\sqrt{\frac{4+3a^2\varepsilon}{4+3a^2\varepsilon z}}. \tag{31}$$

Solving Eq. (15) for $m = 1$, v_1 is obtained as

$$v_1 = \frac{\hbar \varepsilon a^3(9z + 6\varepsilon z a^2 - 1)}{8(4 + 3\varepsilon a^2)}(\cos(3\tau) - \cos(\tau)). \tag{32}$$

Consequently, from the coefficient of cos(τ) in R_2, ω_1 is obtained as follows:

$$\omega_1 = \frac{0.75a^4\varepsilon^2\hbar\omega_e(1 - 10z + 6a^2\varepsilon z^2 - 6a^2\varepsilon z + 9z^2)}{(16 + 12a^2\varepsilon + 12a^2 z\varepsilon + 9a^4 z\varepsilon^2)^{3/2}}. \tag{33}$$

The higher-order approximations for frequency and deflection can be similarly derived. Solving Eq. (15) for $m = 2$ yields the following result for v_2:

$$v_2 = \frac{\hbar \varepsilon a^3(9z + 6\varepsilon z a^2 - 1)}{64(4 + 3\varepsilon z a^2)(4 + 3\varepsilon a^2)^2}(\alpha_1 \cos(\tau) + \alpha_2 \cos(3\tau) + \alpha_3 \cos(5\tau)), \tag{34}$$

where

$$\begin{aligned}\alpha_1 = & -96\varepsilon a^2 - 128\hbar - 96z\varepsilon a^2 - 128 - 92\hbar\varepsilon a^2 - 72z\varepsilon^2 a^4 \\ & - 108\hbar\varepsilon^3 a^6 z^2 - 174\hbar\varepsilon^2 a^4 z^2 - 186\hbar\varepsilon^2 a^4 z - 292\hbar\varepsilon a^2 z,\end{aligned} \tag{35}$$

$$\begin{aligned}\alpha_2 = & \, 128\hbar + 96\hbar\varepsilon a^2 + 192\hbar\varepsilon a^2 z + 96\varepsilon a^2 + 96z\varepsilon a^2 + 72z\varepsilon^2 a^4 \\ & + 128 + 54\hbar\varepsilon^3 a^6 z^2 + 99\hbar\varepsilon^2 a^4 z^2 + 117\hbar\varepsilon^2 a^4 z,\end{aligned} \tag{36}$$

$$\alpha_3 = -4\hbar\varepsilon a^2 + 100\hbar z\varepsilon a^2 + 69\hbar z\varepsilon^2 a^4 + 75\hbar z^2\varepsilon^2 a^4 + 54\hbar z^2\varepsilon^3 a^6. \tag{37}$$

From the coefficient of cos(τ) in R_3, ω_2 is obtained as follows:

$$\omega_2 = \frac{b_1 b_2}{b_3}, \tag{38}$$

where b_1, b_2 and b_3 are given in Appendix A. The [1,1] homotopy Pade approximation of ω and $v(t)$ are written in the following form:

$$\omega_{[1,1]\text{pade}} = \frac{\omega_1\omega_0 + \omega_1^2 - \omega_2\omega_0}{\omega_1 - \omega_2}, \tag{39}$$

$$v_{[1,1]\text{pade}} = \frac{v_1 v_0 + v_1^2 - v_2 v_0}{v_1 - v_2}. \tag{40}$$

Also [m, n] homotopy Pade approximation are determined by the first ($m+n$)th-order approximations of ω and $v(t)$.

4. Results and discussions

In order to demonstrate and verify the accuracy and effectiveness of the HAM, the procedures explained in the previous section are applied to obtain some sets of results, which are presented here. There are many parameters, which can be varied in the governing equation. Table 1 gives the comparison of obtained results with those published in the literature for different m, a, ε, K_1 and K_2.

It can be observed from Table 1 that there is an excellent agreement between the results obtained from the homotopy Pade technique and those reported in the literature. The maximum error between [1,1] homotopy Pade approximant and numerical results (*Runge–Kutta* method) is 0.02 percent. In this case study, HAM provides extremely accurate results for a wide range of system parameters. Also, the vibration frequency converges very quickly to the exact solution with only [1,1] homotopy Pade approximant.

Fig. 2 illustrates the effect of auxiliary parameter $\hbar$ on the frequency for different order approximation of HAM solutions. It is shown that with the increase of the order of approximation, the frequency is independent of $\hbar$ and remains fixed.

Table 1
Comparison of frequency corresponding to various parameters of system

m	a	ε	K_1	K_2	[1,1] Homotopy Pade approximant	Lai and Lim [25]	*Runge–Kutta* [25]
1	0.5	0.5	50	5	2.220231	2.220231	2.220231
1	0.5	0.5	50	5	3.175555	3.175209	3.175501
1	2	0.5	5	5	1.903495	1.900724	1.903569
1	2	0.5	5	50	2.195226	2.194560	2.195284
3	5	1	8	16	1.615021	1.614287	1.615107
3	5	1	10	5	1.748859	1.745984	1.749115
5	10	2	12	16	1.545833	1.545682	1.545853
5	30	5	15	5	1.731378	1.731347	1.731382
10	200	5	5	250	0.707107	0.707107	0.707107
10	100	10	5	25	0.707106	0.707106	0.707106
1	0.5	−0.5	50	5	2.038209	2.038209	2.038209
2	2	−0.1	10	10	1.446365	1.445356	1.446389
3	4	−0.02	30	10	1.318379	1.318255	1.318370
4	10	−0.008	6	3	0.517222	0.514250	0.517327
10	5	−0.01	8	16	0.705406	0.705312	0.705412

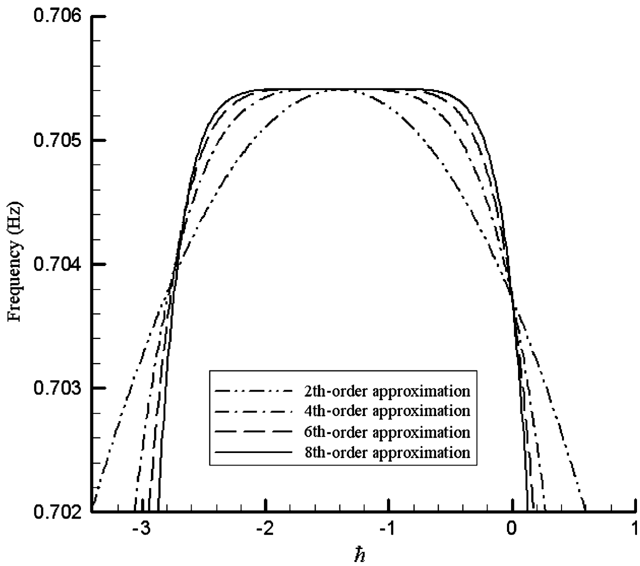

Fig. 2. The effect of auxiliary parameter $\hbar$ on the frequency ($m = 10$, $a = 5$, $\varepsilon = -0.01$, $K_1 = 8$ and $K_2 = 16$).

It can be noted that for $\hbar = -1.408$, which is the middle point of the convergence interval, the second-order approximation gives the frequency with the highest accuracy, while the higher order of approximation (up to the second-order of approximation) does not improve the accuracy of the results. This indicates that the auxiliary parameter plays an important role in the homotopy analysis method.

Fig. 3 presents variation of frequency (ω) versus amplitude associated with the influence of $\varepsilon = (\beta/K_2)$ corresponding to [1,1] homotopy Pade approximant for $m = 3$, $a = 5$, $K_1 = 10$ and $K_2 = 5$.

It can be observed that by increasing the vibration amplitude a, the frequency tends to a constant value independent of ε variation. For the limit in Eq. (39) as the amplitude of vibration tends to infinity, the limiting case of [1,1] homotopy Pade approximant is as follows:

$$\lim_{a\to\infty} \omega_{[1,1]\ \text{pade}} = \frac{\omega_e}{\sqrt{z}}. \tag{41}$$

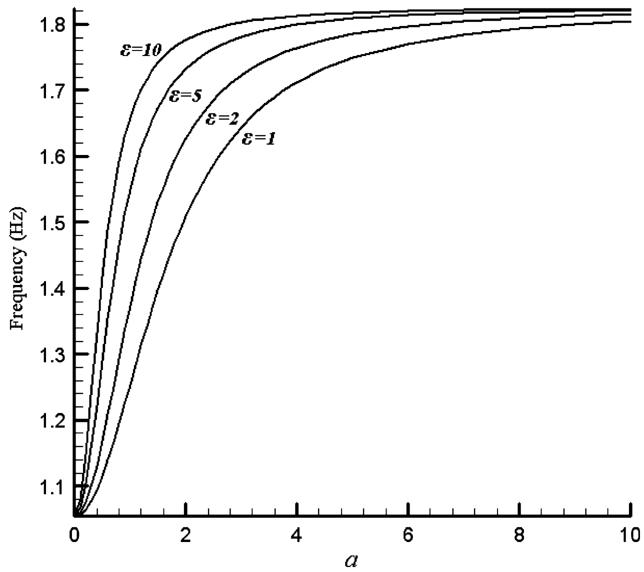

Fig. 3. The results of [1,1] homotopy Pade approximant for natural frequency versus amplitude a associated with the influence of ε.

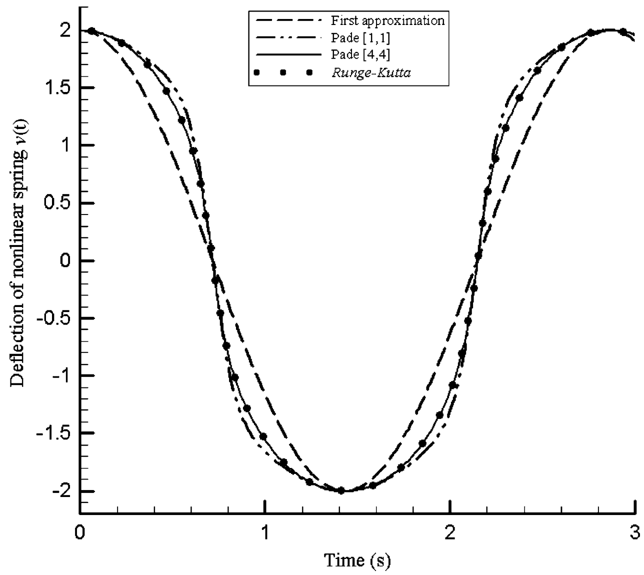

Fig. 4. Deflection of nonlinear spring $v(t)$.

For the limit as the amplitude of vibration tends to infinity, [1,1] homotopy Pade approximant gives identical result to those obtained by Lai and Lim [25].

Figs. 4–6 show the deflection of nonlinear spring $v(t)$, deflection of linear spring $y_1(t)$ and displacement of mass $y_2(t)$, respectively, for $m = 1$, $a = 2$, $\varepsilon = 0.5$, $K_1 = 5$ and $K_2 = 50$.

The accuracy of the [4,4] homotopy Pade approximant is better than the corresponding result of [1,1]. Moreover, the results have an excellent agreement with the numerical solution using *Runge–Kutta* method.

The case study of this paper shows that the HAM and homotopy Pade technique can be potentiality used for the analysis of strongly nonlinear vibration problems with high accuracy. As a significant conclusion, the obtained results show that the accuracy of the present HAM solution is better than other analytical techniques in the considered vibration problem with large oscillation amplitudes. The traditional analytical techniques

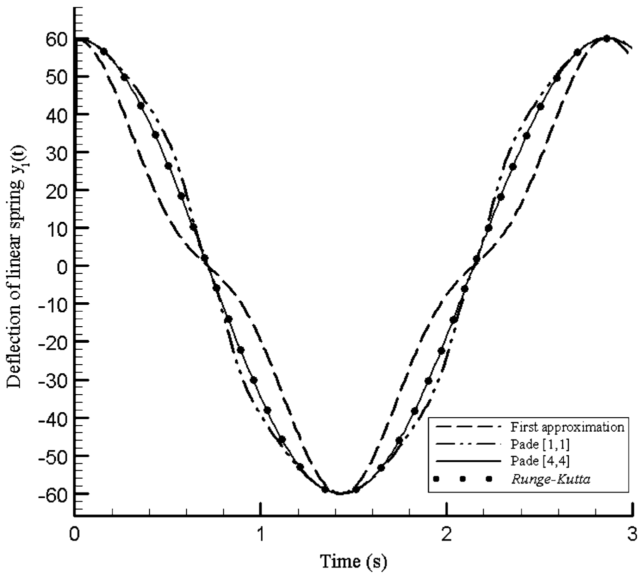

Fig. 5. Deflection of linear spring $y_1(t)$.

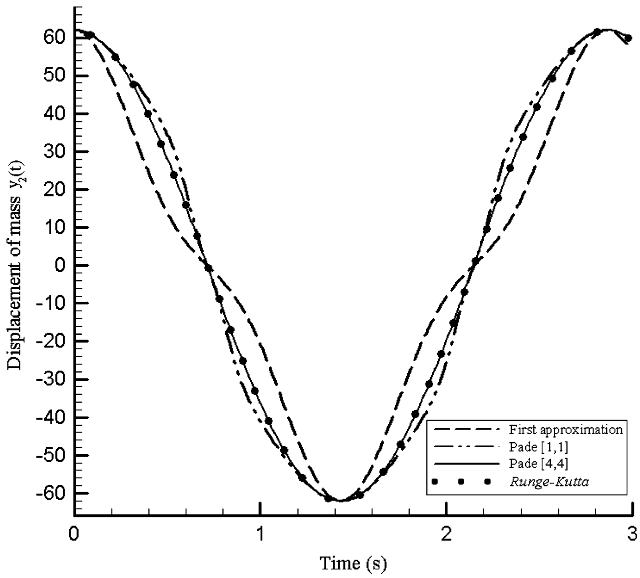

Fig. 6. Displacement of mass $y_2(t)$.

(e.g. perturbation method) loose their reliability at higher amplitudes of vibrations [1]. Also, according to the results, the precision and convergence rate of the solution increases using homotopy Pade approximant.

5. Conclusion

The homotopy analysis method and homotopy Pade technique have been used to obtain an analytical solution for the nonlinear free vibration of a conservative oscillator with inertia and static type cubic nonlinearities. Besides their irreplaceable theoretical value, analytical solutions can also serve as benchmark to check the results of numerical calculations and study various computational methods. A comprehensive parametric study of the dominant parameters (coefficient of nonlinear spring force, linear portion (K_2) of the nonlinear spring constant, linear spring constant (K_1), deflection of nonlinear spring and mass) was carried out. This study shows that the results of HAM and homotopy Pade technique are valid on a wide range of considered system parameters. Moreover, HAM is suitable not only for weak nonlinear problems, but also for strongly nonlinear problems. The most significant features of this method are its excellent accuracy for the whole range of oscillation amplitude values. Also, it can be used to solve other conservative truly nonlinear oscillators with complex nonlinearities.

Appendix A

$$
\begin{aligned}
b_1 &= 0.09375\omega_e \hbar a^4 \varepsilon^2 \sqrt{16 + 12a^2\varepsilon + 12a^2 z\varepsilon + 9a^4 z\varepsilon^2}, \\
b_2 &= 512 - 1296\varepsilon^4 z^2 a^8 - 10\,368\varepsilon z a^2 - 6912\varepsilon^2 z a^4 + 1296\varepsilon^4 z^3 a^8 + 7488\varepsilon^3 z^2 a^4 - 864\varepsilon^2 z^2 a^4 \\
&\quad - 594\hbar\varepsilon^3 z^3 a^6 - 9472\hbar\varepsilon z a^2 - 14\,526\hbar\varepsilon^3 z^2 a^6 - 2816\hbar\varepsilon z^2 a^2 - 17\,946\hbar\varepsilon^2 z^2 a^4 + 15\,540\hbar\varepsilon^2 z^3 a^4 \\
&\quad - 1062\hbar\varepsilon^3 z a^6 + 2592\hbar\varepsilon^5 z^4 a^{10} + 11232\hbar\varepsilon^4 z^4 a^8 - 2592\hbar\varepsilon^5 z^3 a^{10} + 11\,520\hbar\varepsilon z^3 a^2 + 7749\hbar\varepsilon^2 z^4 a^4 \\
&\quad + 16\,182\hbar\varepsilon^3 z^4 a^6 - 7776\hbar\varepsilon^4 z^3 a^8 - 5120z + 512\hbar + 768\varepsilon a^2 + 288\varepsilon^2 a^4 - 5628\hbar\varepsilon^2 z a^4 - 3456\hbar\varepsilon^4 z^2 a^8 \\
&\quad + 4608z^2 - 3888\varepsilon^3 z^2 a^6 + 6144\varepsilon z^2 a^2 + 5400\varepsilon^3 z^3 a^6 + 285\hbar\varepsilon^2 a^4 + 768\hbar\varepsilon a^2 + 3456\varepsilon z^2 a^2 - 1512\varepsilon^3 z a^6 \\
&\quad + 4608\hbar z^2 - 5120\hbar z, \\
b_3 &= 6912a^6 z^3\varepsilon^3 + 11\,664a^{10} z\varepsilon^5 + 2187a^{14} z^3\varepsilon^7 + 11\,664a^{12} z^3\varepsilon^6 + 466\,656a^{10} z^2\varepsilon^5 + 82\,944a^6 z^2\varepsilon^3 + 20\,736a^8 z^3\varepsilon^4, \\
&\quad \times 16\,384 + 124\,416a^6 z\varepsilon^3 + 23\,328a^{10} z^3\varepsilon^5 + 62\,208a^2 z\varepsilon^4 + 36\,864a^2 z\varepsilon + 5184a^8\varepsilon^4 + 27\,648a^4 z\varepsilon^2 + 110\,592a^4 z\varepsilon^2 \\
&\quad \times 8748a^{12} z^2\varepsilon^6 + 27\,648a^4 z^2\varepsilon^2 + 93\,312a^8 z^2\varepsilon^4 + 49\,152a^2\varepsilon + 55\,296a^4\varepsilon^3.
\end{aligned}
$$

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